Bohmnian Trajectories for the Kostin Equation

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Abstract: In this paper we study the Bohmnian Trajectories for the Kostin Equation, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.

Keywords: De Broglie-Bohm Quantum Mechanics; Bohmnian Trajectories of the Kostin Equation

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1. Introduction: The Bohmnian Trajectories

In this article, we calculate the *Bohmnian Trajectories* for the Kostin Equation. To obtain these trajectories we adopted the quantum mechanical formalism of the de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. [1]

2. The Bohmnian Trajectories for the Kostin Equation

Now, let us calculate the Bohmnian trajectories for the Kostin Equation ($K-E$), linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm. [2]

2.1. The Kostin Equation

In 1972, M. D. Kostin [3] proposed a non-linear Schrödinger to represent time dependent physical systems, given by:

$$i\,\eta\,\frac{\partial}{\partial t}\,\psi(x,t) = -\,\frac{\eta^2}{2\,m}\,\frac{\partial^2\psi(x,t)}{\partial x^2} +$$

$$+\,[V(x,t) + \frac{\eta\,\nu}{2\,i}\,\lambda n\,\frac{\psi(x,t)}{\psi^*(x,t)}]\,\psi(x,t)\ , \quad \text{(2.1.1)}$$

where $\psi(x,t)$ and $V(x,t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and ν is a constant.

2.1.1. The Wave Function of the Kostin Equation

Writting the wavefunction $\psi(x,t)$ in the polar form, defined by the Madelung-Bohm [4, 5]:

$$\psi(x,t) = \phi(x,t)\,exp\,[i\,S(s,t)], \quad \text{(2.1.1.1)}$$

where $S(x,t)$ is the classical action and $\phi(x,t)$ will be defined in what follows, and using eq. (2.1.1.1) in eq. (2.1.1), we get: [2]

$$i\,\eta\,(i\,\frac{\partial S}{\partial t} + \frac{1}{\phi}\,\frac{\partial\phi}{\partial t})\,\psi =$$

$$= -\,\frac{\eta^2}{2\,m}\,[i\,\frac{\partial^2 S}{\partial x^2} + \frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2 + 2\,\frac{i}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x}]\,\psi +$$

$$+\,[V(x,t) + \frac{\eta\,\nu}{2\,i}\,\lambda n\,\frac{\phi\,e^{i\,S}}{\phi\,e^{-i\,S}}]\,\psi \quad \rightarrow$$

$$i\,\eta\,(i\,\frac{\partial S}{\partial t} + \frac{1}{\phi}\,\frac{\partial\phi}{\partial t})\,\psi =$$

$$= -\,\frac{\eta^2}{2\,m}\,[i\,\frac{\partial^2 S}{\partial x^2} + \frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2} - (\frac{\partial S}{\partial x})^2 + 2\,\frac{i}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x}]\,\psi +$$

$$+\,[V(x,t) + \eta\,\nu\,S]\,\psi\ . \quad \text{(2.1.1.2)}$$

Taking the real and imaginary parts of eq. (2.1.1.2), we obtain:

a) <u>imaginary part</u>

$$\frac{\eta}{\phi}\,\frac{\partial\phi}{\partial t} = -\,\frac{\eta^2}{2m}\,(\frac{\partial^2 S}{\partial x^2} + \frac{2}{\phi}\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x}) \quad \rightarrow$$

$$\frac{\partial\phi}{\partial t} = -\,\frac{\eta}{2\,m}\,(\phi\,\frac{\partial^2 S}{\partial x^2} + 2\,\frac{\partial S}{\partial x}\,\frac{\partial\phi}{\partial x})\ , \quad \text{(2.1.1.3)}$$

b) <u>real part</u>

$$-\eta\frac{\partial S}{\partial t}=-\frac{\eta^2}{2\,m}[\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}-(\frac{\partial S}{\partial x})^2]+$$

$$+[V(x,t)+\eta\nu S]\quad\rightarrow$$

$$-\frac{\eta}{m}\frac{\partial S}{\partial t}=-\frac{\eta^2}{2\,m^2}\frac{1}{\phi}[\frac{\partial^2\phi}{\partial x^2}-\phi(\frac{\partial S}{\partial x})^2]+$$

$$+\frac{1}{m}[V(x,t)+\eta\nu S]\,.\quad(2.1.1.4)$$

2.1.2. Dynamics of the Kostin Equation

Now, let us see the correlation between eqs. (2.1.1.3,4) and the traditional equations of the Real Fluid Dynamics: [6] a) continuity equation and b) Navier-Stokes's equation. To do is let us perform the following correspondences:

$$\sqrt{\rho(x,t)}=\phi(x,t)\;,\quad(2.1.2.1)\quad\text{(quantum mass density)}$$

$$v_{qu}(x,t)=\frac{\eta}{m}\frac{\partial S(x,t)}{\partial x}\;,\quad(2.1.2.2)\quad\text{(quantum velocity)}$$

$$V_{qu}(x,t)=-\frac{\eta^2}{2\,m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}=-\frac{\eta^2}{2\,m\phi}\frac{\partial^2\phi}{\partial x^2}\,.\quad(2.1.2.3a,b)\quad\text{(Bohm quantum potential)}$$

Putting eq. (2.1.2.1,2) into (2.1.1.3) we get:

$$\frac{\partial\sqrt{\rho}}{\partial t}=-\frac{\eta}{2\,m}(2\frac{\partial S}{\partial x}\frac{\partial\sqrt{\rho}}{\partial x}+\sqrt{\rho}\frac{\partial^2 S}{\partial x^2})\quad\rightarrow$$

$$\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t}=-\frac{\eta}{2\,m}(2\frac{\partial S}{\partial x}\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial x}+\sqrt{\rho}\frac{\partial^2 S}{\partial x^2})\quad\rightarrow$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}=-\frac{\eta}{m}(\frac{\partial S}{\partial x}\frac{1}{\rho}\frac{\partial\rho}{\partial x}+\frac{\partial^2 S}{\partial x^2})\quad\rightarrow$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial x}(\frac{\eta}{m}\frac{\partial S}{\partial x})-\frac{1}{\rho}(\frac{\eta}{m}\frac{\partial S}{\partial x})\frac{\partial\rho}{\partial x}\quad\rightarrow$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}=-\frac{\partial v_{qu}}{\partial x}-\frac{v_{qu}}{\rho}\frac{\partial\rho}{\partial x}\quad\rightarrow$$

$$\frac{\partial\rho}{\partial t}+\rho\,\frac{\partial v_{qu}}{\partial x}+v_{qu}\,\frac{\partial\rho}{\partial x}=0\quad\rightarrow$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho\,v_{qu})}{\partial x}=0\ ,\quad (2.1.2.4)$$

which represents the continuity equation of the mass conservation law of the Fluid Dynamics. [6] We must note that this expression also indicates <u>coerence</u> of the considered physical system represented by (2.1.1).

Now, taking the eq. (2.1.1.4) and using the eqs. (2.1.2.2,3b), will be:

$$-\,\eta\,\frac{\partial S}{\partial t}=-\,(\frac{\eta^2}{2\,m\,\phi})\,\frac{\partial^2\phi}{\partial x^2}+\frac{1}{2}\,m(\frac{\eta}{m}\,\frac{\partial S}{\partial x})^2+$$

$$+[V(x,t)+\eta\,\nu\,S]\ \rightarrow$$

$$\eta\,(\frac{\partial S}{\partial t}+\nu\,S)+(\frac{1}{2}\,m\,v_{qu}^2+V+V_{qu})=0\ .\quad (2.1.2.5)$$

Differentiating the eq. (2.1.1.4) with respect x and using the eqs. (2.1.2.2,3b) we have:

$$-\,\frac{\eta}{m}\,\frac{\partial^2 S}{\partial x\partial t}=-\,\frac{\eta^2}{2\,m^2}\,\frac{\partial}{\partial x}\,[\frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2}-(\frac{\partial S}{\partial x})^2]+\frac{1}{m}\,\frac{\partial V}{\partial x}+\frac{\eta}{m}\,\nu\,\frac{\partial S}{\partial x}\quad\rightarrow$$

$$-\,\frac{\partial}{\partial t}\,(\frac{\eta}{m}\,\frac{\partial S}{\partial x})=\frac{1}{m}\,\frac{\partial}{\partial x}\,(-\,\frac{\eta^2}{2\,m}\,\frac{1}{\phi}\,\frac{\partial^2\phi}{\partial x^2})\ +$$

$$+\ \frac{1}{2}\,\frac{\partial}{\partial x}\,(\frac{\eta}{m}\,\frac{\partial S}{\partial x})^2+\frac{1}{m}\,\frac{\partial V}{\partial x}+\nu\,\frac{\eta}{m}\,\frac{\partial S}{\partial x}\quad\rightarrow$$

$$\frac{\partial v_{qu}}{\partial t}+v_{qu}\,\frac{\partial v_{qu}}{\partial x}+$$

$$+\,\nu\,v_{qu}=-\,\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{qu})\ ,\quad (2.1.2.6)$$

which is an equation similar to the Navier-Stokes's equation which governs the motion of an real fluid.

Considering the "substantive differentiation" (local plus convective) or "hidrodynamic differention": $d/dt=\partial/\partial t+v_{qu}\,\partial/\partial x$ and that $v_{qu}=dx_{qu}/dt$, the eq. (2.1.2.6) could be written as:[6]

$$m\,d^2x/dt^2=-\,\nu\,v_{qu}-\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{qu})\ ,\quad (2.1.2.7)$$

that has a form of the *second Newton law*.

2.1.3. The Quantum Wave Packet of the Linearized Kostin Equation along a Classical Trajetory

In order to find the quantum wave packet of the non-linear Kostin equation, let us considerer the following *ansatz* :[7]

$$\rho\,(x,t) = [2\pi\, a^2(t)]^{-1/2}\, exp\,(-\frac{[x-q(t)]^2}{2\,a^2(t)})\,, \quad (2.1.3.1)$$

where $a(t)$ and $q(t)$ are auxiliary functions of time, to be determined in what follows; they represent the *width* and *center of mass of wave packet*, respectively.

Substituting (2.1.3.1) into (2.1.2.4) and integrated the result, we obtain:[4]

$$v_{qu}\,(x,t) = \frac{\dot{a}(t)}{a(t)}[x-q(t)] + \dot{q}(t)\,, \quad (2.1.3.2)$$

where the integration constant must be equal to zero since ρ and $\rho\,\partial S/\partial x$ vanish for $|x| \to \infty$. In fact, any well-behaved function of (x - X) multiplied by ρ clearly vanishes as $|x| \to \infty$.

To obtain the quantum wave packet of the linear Kostin equation along a classical trajetory given by (2.1.1.1), let us expand the functions $S(X,T)$, $V(x,t)$ and $V_{qu}(x,t)$ around of $q(t)$ up to second Taylor order. In this way we have:

$$S(x,t) = S[q(t),t] + S'[q(t),t]\,[x-q(t)] + \frac{S''[q(t),t]}{2}[x-q(t)]^2\,, \quad (2.1.3.3)$$

$$V(x,t) = V[q(t),t] + V'[q(t),t]\,[x-q(t)] + \frac{V''[q(t),t]}{2}[x-q(t)]^2\,. \quad (2.1.3.4)$$

$$V_{qu}(x,t) = V_{qu}[q(t),t] + V_{qu}'[q(t),t]\,[x-q(t)] + \frac{V_{qu}''[q(t),t]}{2}[x-q(t)]^2\,. \quad (2.1.3.5)$$

Differentiating (2.1.3.3) in the variable x, multiplying the result by $\frac{\eta}{m}$, using the eqs. (2.1.2.2) and (2.1.3.2), taking into account the polynomial identity property and also considering the second Taylor order, we obtain:

$$\frac{\eta}{m}\frac{\partial S(x,t)}{\partial x} = \frac{\eta}{m}\,(\,S'[q(t),t] + S''[q(t),t]\,[x-q(t)]\,) =$$

$$= v_{qu}(x,t) = [\,\frac{\dot{a}(t)}{a(t)}]\,[x_{qu}-q(t)] + \dot{q}(t) \quad \rightarrow$$

$$S'[q(t),t] = \frac{m\,\dot{q}(t)}{\eta}\,,\;\; S''[q(t),t] = \frac{m}{\eta}\frac{\dot{a}(t)}{a(t)}\,, \quad (2.1.3.6a,b)$$

Substituting (2.1.3.6a,b) into (2.1.3.3), results:

$$S(x,t) = S_o(t) + \frac{m\,\dot{q}(t)}{\hbar}[x - q(t)] + \frac{m}{2\,\hbar}\frac{\dot{a}(t)}{a(t)}[x - q(t)]^2 \ , \quad (2.1.3.7)$$

where:

$$S_o(t) \equiv S[q(t), t] \ , \quad (2.1.3.8)$$

are the classical actions.

Differentiating the (2.1.3.7) with respect to t, we obtain (remembering that $\frac{\partial x}{\partial t}$ = 0):

$$\frac{\partial S}{\partial t} = \dot{S}_o(t) + \frac{\partial}{\partial t}\left(\frac{m\,\dot{q}(t)}{\hbar}[x - q(t)]\right) + \frac{\partial}{\partial t}\left(\frac{m}{2\,\hbar}\left[\frac{\dot{a}(t)}{a(t)}\right][x - q(t)]^2\right) \ \to$$

$$\frac{\partial S}{\partial t} = \dot{S}_o(t) + \frac{m\,\ddot{q}(t)}{\hbar}[x - q(t)] - \frac{m\,\dot{q}(t)^2}{\hbar} +$$

$$+ \ \frac{m}{2\,\hbar}\left[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}\right][x - q(t)]^2 - \frac{m\,\dot{q}(t)}{\hbar}\frac{\dot{a}(t)}{a(t)}[x - q(t)] \ . \quad (2.1.3.9)$$

Considering the eqs. (2.1.2.1) and (2.1.3.1), let us write V_{qu} given by (2.1.2.3a,b) in terms of potencies of $[x - q(t)]$. Initially using (2.1.2.1) and (2.1.3.1), we calculate the following derivations:

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left([2\,\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,a^2(t)}}\right) = [2\,\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,a^2(t)}}\,\frac{\partial}{\partial x}\left(-\frac{[x - q(t)]^2}{4\,a^2(t)}\right) \ \to$$

$$\frac{\partial \phi}{\partial x} = -[2\,\pi a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,{}^2(t)}}\,\frac{[x - q(t)]}{2\,a^2(t)} \ ,$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(-[2\,\pi a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4a^2(t)}}\,\frac{[x - q(t)]}{2\,a^2(t)}\right) =$$

$$= -[2\,\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,a^2(t)}}\,\frac{\partial}{\partial x}\left(\frac{[x - q(t)]}{2\,a^2(t)}\right) -$$

$$-[2\,\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,a^2(t)}}\,\frac{\partial}{\partial x}\left(-\frac{[x - q(t)]^2}{4\,a^2(t)}\right) \ \to$$

$$\frac{\partial^2 \phi}{\partial x^2} = -[2\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4a^2(t)}}\,\frac{1}{2\,a^2(t)} + [2\,\pi\,a^2(t)]^{-1/4}\,e^{-\frac{[x-q(t)]^2}{4\,a^2(t)}}\,\frac{[x - q(t)]^2}{4\,a^4(t)} \ =$$

$$= -\phi \, \frac{1}{2\, a^2(t)} + \phi \, \frac{[x-q(t)]^2}{4\, a^4(t)} \quad \rightarrow \quad \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = \frac{[x-q(t)]^2}{4\, a^4(t)} - \frac{1}{2\, a^2(t)} \ . \qquad (2.1.3.10)$$

Substituting (2.1.3.10) into (2.1.2.3b) and taking into account (2.1.3.5), results:

$$V_{qu}(x,t) = V_{qu}[q(t),t] + V_{qu}'[q(t),t]\,[x-q(t)] + \frac{V_{qu}''[q(t),t]}{2}\,[x-q(t)]^2 \ \rightarrow$$

$$V_{qu}(x,t) = \frac{\eta^2}{4\, m\, a^2(t)} - \frac{\eta^2}{8\, m\, a^4(t)}\,[x-q(t)]^2 \ . \qquad (2.1.3.11)$$

Inserting the eqs. (2.1.2.2), (2.1.3.2-4),(2.1.3.7-9) and (2.1.3.11), into (2.1.3.10), we obtain [remembering that $S_o(t)$, $a(t)$ and $q(t)$]:

$$\eta\,[\frac{\partial S}{\partial t} + \nu\, S + (\frac{1}{2}\, m\, v_{qu}^2 + V + V_{qu}) = 0 \ . \qquad (2.1.2.12)$$

$$\eta\,(\,\dot{S}_o + \frac{m\,\ddot{q}(t)}{\eta}\,[x-q(t)] - \frac{m\,\dot{q}^2(t)}{\eta} + \frac{m}{2\,\eta}\,[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2}{a^2(t)}\,]\,[x-q(t)]^2 -$$

$$- \frac{m\,\dot{q}(t)}{\eta}\,\frac{\dot{a}(t)}{a}(t)\,[x-q(t)]\,) + \frac{m}{2}\,(\,\frac{\dot{a}(t)}{a(t)}\,[x-q(t)] + \dot{q}(t)\,)^2 +$$

$$+ \eta\,\nu\,(\,S_0(t) + \frac{m\,\dot{q}(t)}{\eta}\,[x-q(t)] + \frac{m}{2\,\eta}\,\frac{\dot{a}(t)}{a(t)}\,[x-q(t)]^2\,) +$$

$$+ V[q(t),t] + V'[q(t),t]\,[x-q(t)] + \frac{1}{2}\,V''[q(t),t]\,[x-q(t)]^2 \ +$$

$$+ \frac{\eta^2}{4\, m\, a^2(t)} - \frac{\eta^2}{8\, m\, a^4(t)}\,[x-q(t)]^2 = 0 \ . \qquad (2.1.3.13)$$

Since $(x-q)^o = 1$, expanding (2.1.3.13) in potencies of $(x-q)$, we obtain:

$$(\,\eta\,\dot{S}_o(t) - m\,\dot{q}^2(t) + \frac{1}{2}\,m\,\dot{q}^2(t) + V[q(t),t] + \eta\,\nu\,[S_0(t)] + \frac{\eta^2}{4\, m a^2(t)}\,)\,[x-q(t)]^o +$$

$$+ (\,m\,\ddot{q}(t) - m\,\dot{q}(t)\,\frac{\dot{a}(t)}{a(t)} + m\,\dot{q}(t)\,\frac{\dot{a}(t)}{a(t)} + \nu\,m\,\dot{q}(t) + V'[q(t),t])\,[x-q(t)] \ +$$

$$+ (\,\frac{m}{2}\,[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}(t)^2}{a(t)^2}] + \frac{m}{2}\,\frac{\dot{a}^2(t)}{a^2(t)} + m\,\nu\,\frac{\dot{a}(t)}{a(t)} +$$

$$+\frac{1}{2}V''[q(t),t] - \frac{\eta^2}{4\,ma^4(t)}\,)\,[x-q(t)]^2 = 0\ . \quad (2.1.3.14)$$

As (2.1.3.14) is an identically null polynomium, all coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\eta}\,(\,\frac{1}{2}\,m\,\dot{q}(t)^2 - V[q(t),t] - \frac{\eta^2}{4\,m\,a(t)^2} - \eta\,\nu\,S_0(t))\ , \quad (2.1.3.15)$$

$$\ddot{q}(t) + \nu\,\dot{q}(t) + \frac{1}{m}V'[q(t),t] = 0\ , \quad (2.1.3.16)$$

$$\ddot{a}(t) + \nu\,\dot{a}(t) + \frac{a(t)}{m}V''[q(t),t] - \frac{\eta^2}{4\,m^2\,a^3(t)} = 0\ . \quad (2.1.3.17)$$

Now, let us consider that V[q(t), t] is given by:

$$V[q(t),t] = \frac{1}{2}\,m\,\omega^2(t)\,q^2(t)\,, \quad (2.1.3.18)$$

which is the Time Dependent Harmonic Oscillator Potencial.
In this case, we have:

$$V'[q(t),t] = m\,\omega^2(t)\,q(t), \quad V''[q(t),t] = m\,\omega^2(t)\ . \quad (2.1.3.19a,b)$$

Putting the eqs. (2.1.3.19a,b) into eqs. (2.1.3.16, 17), results:

$$\ddot{q}(t) + \omega^2(t)\,q(t) = 0\ , \quad (2.1.3.20)$$

$$\ddot{a}(t) + \nu\,\dot{a}(t) + a(t)\omega^2(t) = \frac{\eta^2}{4\,m^2\,a^3(t)}\ . \quad (2.1.3.21)$$

2.1.4. The Bohmnian Trajectories for the Kostin Equation

The associated Bohmnian Trajectories, [8-12] for the Kostin Equation ($K-E$) of an evolving *ith* particle of the ensemble with an initial position x_{0i} can be calculated by considering that:

$$\dot{x}_i(t) = v_{qu}[x_i(t),t]\,. \quad (2.1.4.1)$$

Then substituting the eq. (2.1.4.1) into eq. (2.1.3.2), results:

$$\dot{x}_i(t) = \frac{\dot{a}(t)}{a(t)}\times[x_i(t)-q(t)] + \dot{q}(t) \ \to\ \frac{\dot{x}_i(t)-\dot{q}(t)}{x_i(t)-q(t)} = \frac{\dot{a}(t)}{a(t)} \ \to$$

$$\int_0^t \frac{d}{dt}\{\,\lambda n\,[x_i(t)-q(t)]\,\}\,dt = \int_0^t \frac{d}{dt}\{\,\lambda n\,[a(t)]\,\}\,dt \ \to$$

$$\lambda n \left\{ \frac{[x_i(t) - q(t)]}{[x_{0i} - q_0]} \right\} = \lambda n \left\{ \frac{a(t)}{a_0} \right\} \;\; \rightarrow \;\; \frac{[x_i(t) - q(t)]}{[x_{0i} - q_0]} = \frac{a(t)}{a_0} \;\; \rightarrow$$

$$x_i(t) = q(t) + (x_{0i} - q_0) \times \frac{a(t)}{a_0} . \quad (2.1.4.2)$$

The eqs. (2.1.3.20,21) show that a continuous measurement of a quantum wawe packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consist of monitoring the position of quantum systems and the result is the measured classical path $q(t)$ for t within a quantum uncertainty $a(t)$.

2.1.4.1. The Bohmnian Trajectories for the Kostin Equation

From the eqs. (2.1.3.20,21), we note that for $\nu \neq 0$ a stationary regime can be reached and that the width [$a(t)$] of the wave packet can be related to the resolution of measurement as follows. Then considering that $a(t) = cte$ [$\dot{a}(t) = 0$] in the eqs. (2.1.3.21), we have:

$$a_0 \; \omega_0^2 = \frac{\eta^2}{4\, m^2 a_0^3} \rightarrow \omega_0 = \frac{1}{\tau_B} \;\; , \quad (2.1.4.1.1a)$$

where [8, 13]:

$$\tau_B = \left(\frac{2\, m\, a_0^2}{\eta} \right) = 6{,}8 \times 10^{-26} s \, , \quad (2.1.4.1.1b)$$

is the *Bohmtime constant* which determines the time resolution of the quantum measurement, and:

$$\ddot{q}(t) + \omega^2(t)\, q(t) = 0 \;\; \rightarrow \;\; q(t) = q_0 \; exp(\pm\, i\; \omega_0\; t) \; . \quad (2.1.4.1.2)$$

3. Conclusion

The eqs. (2.1.4.1a,b) means that if an initially free wave packet is kept under a certain continuous measurement, its (a_0) may not spread in time. Then, the associated *Bohmnian Trajectories* [eq. (2.1.4.1.2)] of an evolving *ith* particle of the ensemble with an initial position x_{0i} is giving by:

$$x_i(t) = q_0 \times exp\; (\pm\, \omega_0\; t) + (x_{0i} - q_0) \times \frac{a(t)}{a_0} \; . \quad (2.1.4.1.3)$$

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13. If the initial wave packet width (a_0) is taken to be equal to $2{,}8 \times 10^{-15}\, m$ (the approximate size of an electron of mass m) then τ_B to be about 10^{-25} sec, for a continuous measurement. We note that (see 8.), experiments to measure the size of the electron consist on colliding two beams of electrons against each other and counting how many are scattered and altered their trajectories. By counting the collisions, and knowing how many particles we have thrown, we can estimate the average size of each particle in the beam.